Lie Groups and Lie Algebras

Lecture II: Hilbert's Fifth and the Friends We'll Make Along the Way

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Oh, The Places We'll Go!

What is a manifold?

The intuitive understanding of a manifold is a set that:

- 'Nice' topological spaces
- Locally look like euclidean space (corresponding to their dimension)
- A differentiable structure allows us to do 'calculus'

What is a manifold?

The formal definition of 'locally Euclidean':

Definition

1. A *locally Euclidean space* M^d *of dimension* d *is a Hausdorff topological space* M^d *for which each point has a neighborhood homeomorphic to an open subset of Euclidean space* $\mathbb{R}^d$

What is a manifold?

The formal definition of 'can do calculus':

Definition

2. A *differentiable structure* $\mathscr{F}$ of class $C^k (1 \leq k \leq \infty)$ on a locally Euclidean space M is a collection of coordinate systems $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ satisfying the following three properties:

(a) $\displaystyle\bigcup_{\alpha \in A} U_\alpha = M.$

(b) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is C^k for all $\alpha, \beta \in A$.

(c) The collection $\mathscr{F}$ is maximal with respect to (a) and (b).

What is a manifold?

We gave the definition of a manifold:

Definition

A *d-dimensional differential manifold of class* C^k is a pair $(M, \mathscr{F})$ comprised of a d-dimensional, second countable, locally Euclidean space M together with a differentiable structure $\mathscr{F}$ of class C^k.

$(M, \mathscr{F})$ will usually be shortened to just M with $\mathscr{F}$ being implicit.

Examples of Manifolds

Example

- $\mathbb{R}$ with the binary operation $+$
- $\mathbb{C}^{\times}$ under multiplication.
- $\mathbb{T} = (\mathbb{S}^1 \times \mathbb{S}^1); \mathbb{T}^n$

The Differential

Being able to do calculus on our manifold, we generalized the familiar derivative to manifolds:

Definition

Let $\psi : M \to N$ be C^∞, and let $p \in M$. The *differential of* ψ *at* p is the linear map

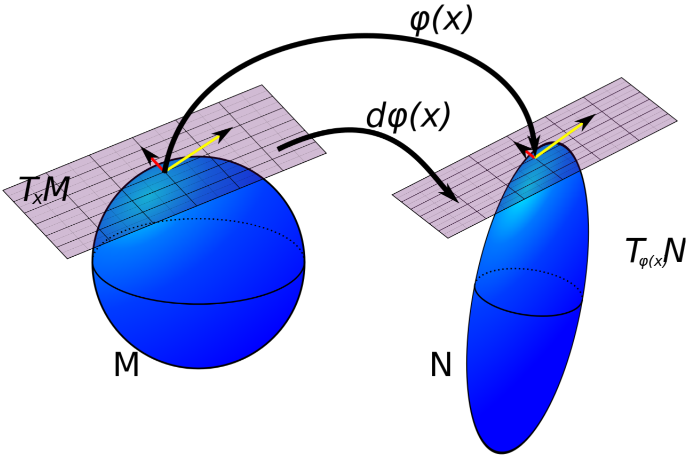
$$d\psi : M_p \to N_{\psi(p)}$$

defined by

$$d\psi(v)(g) = v(g \circ \psi)$$

where v is a tangent vector at p and g is a C^∞ near $\psi(p)$. We say ψ is *non-singular* if $d\psi$ is injective.

The Differential



Steeped in Subobjects

With the definition of the differential we defined immersions, submanifolds, and embeddings:

Definition

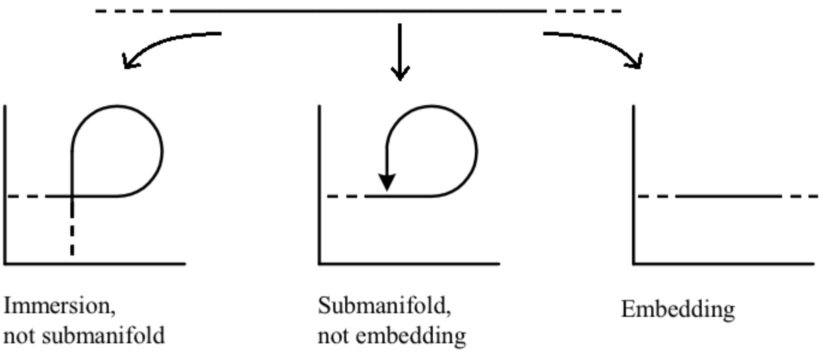
Let $\psi : M \to N$ be C^∞.
(a) ψ is an *immersion* if $d\psi_p$ is injective for each $p \in M$.

(b) $\star$ The pair (M, ψ) is a *submanifold* of N if ψ is an injective immersion.

(c) ψ is an imbedding if it is a homeomorphic immersion.

(d) ψ is a *diffeomorphism* if ψ is a bijection from M onto N and ψ^{-1} is C^∞.

Steeped in Subobjects



Vector Fields

We also generalized vector fields to our manifolds:

Definition

A *vector field on an open set U in M* is a lifting of U into TM, that is, a map $X : U \to TM$ such that $\pi \circ X = id_U$.

If X is a vector field on U and $p \in U$, then $X(p)$ (which will be denoted X_p) is an element of M_p. If f is a smooth function on U, then $X(f)$ is the function $X(f)(p) = X_p(f)$ for $p \in U$.

Wasn't this a talk on Lie groups?

We defined a Lie group as:

Definition

A *Lie group* is a differential manifold with a binary operation on its elements such that the map $(\sigma, \tau) \mapsto \sigma\tau^{-1}$ is smooth.

Example

- The manifold $GL(n)$ of $n \times n$ non-singular matrices is a Lie group under matrix multiplication. Many of the 'classical groups' are subgroups of $GL(n, \mathbb{F})$
- $SO(n) \subset O(n)$ with determinate 1.
- The group $U(n)$ of unitary matrices such that $A^{-1} = A^*$.
- $\{z \in \mathbb{C} : |z| = 1\} = U(1) \cong SO(2) \cong \mathbb{T}^1 \cong \mathbb{R}/\mathbb{Z}$.

Topological Groups

Definition

A topological group is a group $G = (G, \cdot)$ that is also a topological space with a binary operation on its elements such that the map $(\sigma, \tau) \mapsto \sigma\tau^{-1}$ is continuous.

Contrast this definition with the definition of a Lie group.

Topological Groups

The group structure bestows 'powered-up' properties of the topological space. For instance:

Properties of Topological Groups

- G is $T_0 \iff G$ is completely regular.
- A subgroup H of G is open if and only if it is closed as $H^c = \bigcup_{g \in G \setminus H} gH$.

Hilbert's Fifth Problem

Last time we teased Hilbert's Fifth Problem and an alternate construction of Lie groups. We give it a proper treatment now. Recall that Hilbert's fifth problem asks thus:

Question: Hilbert's Fifth

Let G be an topological group that is locally Euclidean. Does it follow that G is isomorphic to a Lie group?

Answer: 'Yes', proved by Montgomery-Zippin and Gleason in 1952.

Hilbert's Fifth Problem

The point of Hilbert's Fifth problem is weakening the assumptions of a Lie group and asking if the properties are still strong enough to generate the smooth structure. Locally Euclidean can be hard to work with, though local compactness is more tractable:

Definition

A topological space X is *locally compact* if for every point $x \in X$ there exists an open set U and a compact set K such that $x \in U \subseteq K$.

Hilbert's Fifth Problem

Question: What about weakening the hypotheses of Hilbert's fifth? Is locally compact enough?

Counterexample: The Infinite-Dimensional Torus

The torus $\mathbb{T}$ is a compact (abelian) Lie group. As the product of two Lie groups is a Lie group, $\mathbb{T}^n$ is as well. $\mathbb{T}^\omega$ is compact by Tychonoff's theorem, but is not a Lie group.

Gleason-Yamabe Theorem

The idea can be salvaged though:

Theorem (Gleason-Yamabe Theorem)

Let G be a locally compact topological group. Then for every open neighborhood U of the identity, there exists a subgroup G' of G and a compact normal subgroup K of G' with the following properties:

1. *G' is an open subgroup of G, and K is contained in U.*
2. *G'/K is isomorphic to a Lie group.*

Gleason-Yamabe Theorem

What's the idea? That these locally compact groups are Lie groups in the 'mid-range'. Recall that open subgroups of a topological group G are also closed. If G is connected, then G is the only open subgroup of G.

No Small Subgroups

Definition

We say a topological group G has the *no small subgroup (NSS) property* if there exists an open neighborhood of the identity that contains no non-trivial subgroups of G.

Applying the Gleason-Yamabe theorem to a topological group with NSS then lets us take K to be trivial.

Theorem

All connected locally compact groups with NSS are Lie groups.

Lie Algebras

Definition

A *Lie algebra* $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a bilinear operator $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the bracket) such that such that for all $x, y, z \in \mathfrak{g}$,

(a) $[x, y] = -[y, x]$ (*anti-commutativity*)

(b) $\big[[x, y], z\big] + \big[[y, z], x\big] + \big[[z, x], y\big] = 0$ (*Jacobi identity*)

Left Invariant Vector Fields

We defined what it means for vector fields to be left invariant:

Definition

A vector field X (not assumed apriori to be smooth) on G is called *left invariant* if for each $\sigma \in G$, X is ℓ_σ-related to itself; that is,

$$d\ell_\sigma \circ X = X \circ \ell_\sigma.$$

The set of all left invariant vector fields on a Lie group G will be denoted by the corresponding lowercase German letter $\mathfrak{g}$.

Left Invariant Vector Fields

The following is an important identification of a Lie algebra and the tangent space to a Lie group at the identity:

Theorem

Let G be a Lie group and $\mathfrak{g}$ its set of left invariant vector fields.
(a) $\mathfrak{g}$ is a real vector space, and the map $\alpha : \mathfrak{g} \to G_e$ defined by $\alpha(X) = X(e)$ is an isomorphism of $\mathfrak{g}$ with the tangent space G_e to G at the identity. Consequently, $\dim \mathfrak{g} = \dim G_e = \dim G$.
(b) Left invariant vector fields are smooth.
(c) The Lie bracket of two left invariant vector fields is itself a left invariant vector field.
(d) $\mathfrak{g}$ forms a Lie algebra under the Lie bracket operation on vector fields.

Lie Algebra of a Lie Group

Definition

Define the *Lie algebra of the Lie group* G to be the Lie algebra $\mathfrak{g}$ of the left invariant vector fields on G.
Equivalently, the *Lie algebra of* G is the tangent space G_e with Lie algebra structure furnished by the isomorphism (of Lie algebras) we just proved.

Example

The left invariant vector fields (and now by definition, the Lie algebra) of $\mathbb{R}$ are the constant vector fields $\{\lambda \frac{d}{dr} : \lambda \in \mathbb{R}\}$.

Homomorphisms

We reiterate the definition of a (Lie-group) homomorphism:

Definition

A map $\varphi : G \to H$ is a *(Lie group) homomorphism* if φ is both C^∞ and a group homomorphism of the abstract groups. We call φ an *isomorphism* if, in addition, φ is a diffeomorphism.

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a map $\psi : \mathfrak{g} \to \mathfrak{h}$ is a *(Lie algebra) homomorphism* if it is linear and preserves brackets ($\psi[X, Y] = [\psi(X), \psi(Y)]$ for all $X, Y \in \mathfrak{g}$). If ψ is also bijective, then it is an *isomorphism*.

Lie Subgroups

How we want to define a Lie subgroup is similar to how we defined a submanifold:

Definition

We define (H, φ) to be a *Lie subgroup* of the Lie group G if:

(a) H is a Lie group;

(b) (H, φ) is a submanifold of G;

(c) $\varphi : H \to G$ is a (Lie group) homomorphism.

If, in addition, $\varphi(H) \subset_{c\ell} G$, then (H, φ) is called a *closed subgroup of* G.

The Exponential Map

Definition

A homomorphism $\varphi : \mathbb{R} \to G$ is called a *1-parameter subgroup* of G.

Definition

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Let $X \in \mathfrak{g}$. Then

$$\lambda \frac{d}{dr} \mapsto \lambda X$$

is a homomorphism of the Lie algebra of $\mathbb{R}$ into $\mathfrak{g}$.

The Lie group-Lie Algebra Correspondence

Theorem

3. Let G and H be connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively and with G simply connected. Let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism. Then there exists a unique homomorphism $\varphi : G \to H$ such that $d\varphi = \psi$.

Corollary

If simply connected Lie groups G and H have isomorphic Lie algebras, then G and H are isomorphic.

The Exponential Map

Statement: 3. Let G and H be connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively and with G simply connected. Let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism. Then there exists a unique homomorphism $\varphi : G \to H$ such that $d\varphi = \psi$.

We elucidate this theorem by applying it: $\mathbb{R}$ is simply connected, so there exists a unique 1-parameter subgroup

$$\exp_X : \mathbb{R} \to G$$

such that

$$d\exp_X\left(\lambda \frac{d}{dr}\right) = \lambda X.$$

The Exponential Map

That is, $t \mapsto \exp_X(t)$ is the unique 1-parameter subgroup of G whose tangent vector at 0 is $X(e)$.

Definition

Define the *exponential map* $\exp : \mathfrak{g} \to G$ by setting $\exp(X) = \exp_X(1)$.

The other ways to define the exponential map can also be used, though some are unwieldy for our intentions.

Properties of the Exponential Map

Theorem (Properties of the Exponential Map)

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G and $X \in \mathfrak{g}$. Then

(a) $\exp(tX) = \exp_X(t)$ *for each* $t \in \mathbb{R}$.

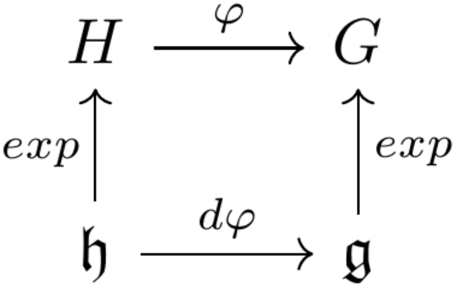
(b) $\exp(t_1 + t_2)X = (\exp_{t_1})(\exp_{t_2})(X)$ *for all* $t_1, t_2 \in \mathbb{R}$.

(c) $\exp(-tX) = (\exp_X(t))^{-1}$ *for each* $t \in \mathbb{R}$.

(d) $\exp : \mathfrak{g} \to G$ *is* C^∞ *and* $d\exp : \mathfrak{g}_0 \to G_e$ *is the identity map, so exp furnishes a diffeomorphism of a neighborhood of 0 in* $\mathfrak{g}$ *onto a neighborhood of* $e \in G$.

Theorem

Let $\varphi : H \to G$ be a homomorphism. Then the follow diagram commutes:



The Power of Structure

Theorem

Let $\varphi : \mathbb{R} \to G$ be a continuous homomorphism of $\mathbb{R}$ in the Lie group G. Then φ is C^∞.

Theorem

Let $\varphi : H \to G$ be a continuous homomorphism of Lie groups. Then φ is C^∞.

The Power of Structure

Theorem

Let $\varphi : H \to G$ be a continuous homomorphism of Lie groups. Then φ is C^∞.

Proof.

Let H be dimension d, and let $X_1, \ldots, X_d$ be a basis of $\mathfrak{h}$. Define the map $\alpha : \mathbb{R}^d \to H$ by

$$\alpha(t_1, \ldots, t_d) = \big(\exp(t_1 X_1)\big) \cdots \big(\exp(t_d X_d)\big).$$

By the fourth part of the properties of the exponential map, α is C^∞ and non-singular at $0 \in \mathbb{R}^d$.

Theorem

Let $\varphi : H \to G$ be a continuous homomorphism of Lie groups. Then φ is C^∞.

Proof.

Then let V be a neighborhood of $0 \in \mathbb{R}^d$ that is diffeomorphic to a neighborhood U of $e \in H$ under α. Thus we have a continuous homomorphism $t \mapsto \varphi\big(\exp(tX_j)\big)$ is a continuous homomorphism of $\mathbb{R}$ into G, and so we can conclude that it is C^∞ by the preceding theorem. Then $\varphi \circ \alpha$ is C^∞ as well, so $\varphi|_U = (\varphi \circ \alpha) \circ \alpha^{-1}|_U$ is C^∞. Since $\varphi|_{hU} = \ell_{\varphi(h)} \circ \varphi \circ \ell_{(h)^{-1}}|_{hU}$, φ is C^∞ on all of H.

Theorem

Let $\varphi : H \to G$ be a continuous homomorphism of Lie groups. Then φ is C^∞.

Proof.

To see that φ is C^∞ on all of H, let $h \in H$ and $u \in U$. Then:

$$\begin{aligned}\varphi(hu) &= \varphi(h)\varphi(u)\\ &= \ell_{\varphi(h)}\big(\varphi(u)\big)\\ &= \big(\ell_{\varphi(h)} \circ \varphi(u)\big)(h^{-1}hu)\\ &= \big(\ell_{\varphi(h)} \circ \varphi \circ \ell_{(h)^{-1}}\big)(u).\end{aligned}$$

Hence $\varphi|_{hU} = \ell_{\varphi(h)} \circ \varphi \circ \ell_{(h)^{-1}}|_{hU}$, so φ is C^∞ on all of H. ∎

The Power of Structure

Theorem

Let $\varphi : H \to G$ be a continuous homomorphism of Lie groups. Then φ is C^∞.

Corollary

A second countable locally Euclidean topological group can have at most one differentiable structure making it into a Lie group.

The Coveted Closed Subgroups

We conclude with an important class of subgroups:

Theorem

Let G be a Lie group and A be a closed abstract subgroup of G. Then A has a unique structure which makes A into a Lie subgroup of G.

The topology imposed on A is indeed what you think it is: the subspace topology.

The Coveted Closed Subgroups

Example

We can apply this instantly:

- The manifold $GL(n)$ of $n \times n$ non-singular matrices is a Lie group under matrix multiplication. Many of the 'classical groups' are subgroups of $GL(n, \mathbb{F})$
- $SL(n)$, the group of $n \times n$ matrices with determinate 1.
- The group $U(n)$ of unitary matrices such that $A^{-1} = A^*$.
- $O(n)$, the group of $n \times n$ matrices such that $A^{-1} = A^T$.
- $SO(n) = O(n) \cap SL(n)$

Our Tools and Our Heading

Let's take shop. We can form new Lie groups from old ones by:

- Taking the product of two Lie groups.
- Taking the quotient of a Lie group by a closed normal subgroup.
- Investigating the universal cover of a connected Lie group.
- Determining the closed abstract subgroups of a Lie group.

In the conclusion to this series we wrap up our investigation of Hilbert's Fifth problem, give an alternate definition of a Lie group, and discuss p-adic Lie groups.

Thank you!
Have a good week!